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VIRTUAL MASS IN MULTIPHASE FLOW

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Abstract—Virtual mass terms for multiphase flow are derived from a three-field representation of bubbles, bulk liquid and the liquid associated with each bubble.

INTRODUCTION

In the two-phase motion of a bubble through a liquid, the total effective mass of the bubble consists of the mass of the vapor itself plus a virtual mass that arises from the inertial properties of the liquid in the immediate vicinity of the bubble. The virtual mass for a single bubble has been derived by Lamb (1952), Prandtl (1952) and Milne-Thomson (1968) for simple bubble shapes and slow motion through an unbounded, inviscid fluid. A discussion of virtual mass effects has also been presented by Thomas *et al.* (1983). In this classical case none of the surrounding fluid is trapped by the bubble, but instead is displaced during its passage. Our goal is to extend previous derivations to obtain a reasonable representation of the virtual mass effect for relatively high speed multiphase flow.

Formulations of the virtual mass term introduce a coefficient, C_{vm}, which describes the volume of displaced fluid that contributes to the effective mass of the bubble. We introduce a similar coefficient, which plays an analogous role in the analysis. As shown by Drew *et al.* (1979), the generalizations proposed by Hinze (1962), Soo (1967) and Wallis (1969) lack mathematical objectivity. In extending the formulations to a representation that is mathematically objective, Drew *et al.* (1979) introduce an undetermined parameter, which is avoided by the approach described below.

Our approach is to commence with a three field formulation representing the bulk liquid as field 1, the bubble vapor as field 2, and the surrounding liquid associated with the virtual mass inertia as field 3. The essence of our derivation lies in the assumption that fields 2 and 3 are very strongly coupled together in the relatively complicated flow circumstance that we wish to describe. In this respect the material of field 3 is dynamically entrapped by the bubble. We assume the entrapped fluid occupies a volume fraction that is a fixed factor, f, of the volume fraction of the bubbles. With suitable descriptions for the relationships among the three fields, that is, the bulk liquid, the bubbles, and the entrapped liquid, our formulation reduces to the usual two-field equations but with a new representation for the virtual mass terms. Mathematical objectivity is ensured by the nature of the initial three-field formulation. Generalization to the case of variable f is possible in principle but requires a detailed examination of the microphysics, especially for circumstances involving nearly equal volume fractions for the bubbles and liquid. Our results are also applicable to droplets and particles. For simplicity we present the derivation for the case in which both the liquid and gas are incompressible.

THREE-FIELD FLOW

We denote the liquid phase by subscript 1 and the gas phase by subscript 2. The starting point for the derivation, however, splits the liquid into two separate fields, the bulk liquid retaining the subscript 1 and adding a prime, and the entrapped liquid being denoted by

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subscript 3. The equations that express mass conservation are

$$\frac{\partial \rho_1 \alpha_1'}{\partial t} + \nabla \cdot \rho_1 \alpha_1' \boldsymbol{u}_1' = 0, \tag{1}$$

$$\frac{\partial \rho_2 \alpha_2}{\partial t} + \nabla \cdot \rho_2 \alpha_2 \boldsymbol{u}_2 = 0, \tag{2}$$

and

$$\frac{\partial \rho_1 \alpha_3}{\partial t} + \nabla \cdot \rho_1 \alpha_3 \boldsymbol{u}_3 = 0. \tag{3}$$

In the equations ρ is the microscopic material density; α, the volume fraction; and $\boldsymbol{u}$, the velocity.

Momentum conservation is described by

$$\frac{\partial \rho_1 \alpha_1' \boldsymbol{u}_1'}{\partial t} + \nabla \cdot (\rho_1 \alpha_1' \boldsymbol{u}_1' \boldsymbol{u}_1') = -\alpha_1' \nabla p + \rho_1 \alpha_1' \boldsymbol{g} + K_{12}(\boldsymbol{u}_2 - \boldsymbol{u}_1') + K_{13}(\boldsymbol{u}_3 - \boldsymbol{u}_1') + \rho_1 \boldsymbol{V}_1, \tag{4}$$

$$\frac{\partial \rho_2 \alpha_2 \boldsymbol{u}_2}{\partial t} + \nabla \cdot (\rho_2 \alpha_2 \boldsymbol{u}_2 \boldsymbol{u}_2) = -\alpha_2 \nabla p + \rho_2 \alpha_2 \boldsymbol{g} + K_{12}(\boldsymbol{u}_1' - \boldsymbol{u}_2) + K_{23}(\boldsymbol{u}_3 - \boldsymbol{u}_2) + \rho_2 \boldsymbol{V}_2, \tag{5}$$

and

$$\frac{\partial \rho_1 \alpha_3 \boldsymbol{u}_3}{\partial t} + \nabla \cdot (\rho_1 \alpha_3 \boldsymbol{u}_3 \boldsymbol{u}_3) = -\alpha_3 \nabla p + \rho_1 \alpha_3 \boldsymbol{g} + K_{13}(\boldsymbol{u}_1' - \boldsymbol{u}_3) + K_{23}(\boldsymbol{u}_2 - \boldsymbol{u}_3) + \rho_1 \boldsymbol{V}_3. \tag{6}$$

The pressure p is assumed to be locally the same for all three phases; $\boldsymbol{g}$ is the acceleration due to gravity. The quantities $\rho_1 \boldsymbol{V}_1$, $\rho_2 \boldsymbol{V}_2$ and $\rho_1 \boldsymbol{V}_3$ represent the transport of momentum due to the viscosity of the respective fields. These terms can be written as the divergence of a tensor, $\rho \boldsymbol{V} \equiv \nabla \cdot \boldsymbol{F}_V$. The interfacial momentum exchange functions are K_{12}, K_{13} and K_{23}, respectively, which may vary with the relative velocities, the volume fractions, and other relevant variables, so that our formulation is by no means limited to a linear model for the exchange of momentum. We have not however included the effects of turbulence or the possibility for non-isotopic momentum exchange.

To complete the three-field set we write

$$\alpha_1' + \alpha_2 + \alpha_3 = 1. \tag{7}$$

RELATIONSHIPS BETWEEN FIELDS

The two-field set of equations is obtained by recognizing that

$$\alpha_1 = \alpha_1' + \alpha_3, \tag{8}$$

and

$$\alpha_1 \boldsymbol{u}_1 = \alpha_1' \boldsymbol{u}_1' + \alpha_3 \boldsymbol{u}_3. \tag{9}$$

The second equation states that the total momentum of the liquid phase is the sum of the two separate momenta, contained in the bulk and entrapped fields. Our primary assumption is that K_{23} is very large so that we may approximate the dynamics of field 3 by writing

$$\mathbf{u}_3 = \mathbf{u}_2 . \quad [10]$$

In addition, we utilize a factor f analogous to C_{vm} such that

$$\alpha_3 = f\alpha_2 . \quad [11]$$

In general f is a function of such flow properties as bubble shape and relative velocity, but for this derivation we have assumed f is a constant. These equations can be combined to show that

$$\alpha'_1 \mathbf{u}'_1 = \alpha_1 \mathbf{u}_1 - f\alpha_2 \mathbf{u}_2 , \quad [12]$$

and

$$\alpha'_1 = \alpha_1 - f\alpha_2 . \quad [13]$$

TWO-PHASE REPRESENTATION

Using [10]–[13], we reduce the three-field mass equations to

$$\frac{\partial \alpha_1}{\partial t} + \nabla \cdot \alpha_1 \mathbf{u}_1 = 0 , \quad [14]$$

and

$$\frac{\partial \alpha_2}{\partial t} + \nabla \cdot \alpha_2 \mathbf{u}_2 = 0 . \quad [15]$$

Likewise the momentum equations can be arranged in such a way as to eliminate the field 3 and prime variables. Add [5] and [6], substitute for the prime variables and multiply by $\rho_2/(\rho_2 + f\rho_1)$ to obtain

$$\rho_2\left[\frac{\partial \alpha_2 \mathbf{u}_2}{\partial t} + \nabla \cdot \alpha_2 \mathbf{u}_2 \mathbf{u}_2\right] = -\rho_2 \alpha_2 \frac{(1+f)}{(\rho_2 + f\rho_1)} \nabla p + \frac{\rho_2 \alpha_1 (K_{12} + K_{13})}{(\rho_2 + f\rho_1)(\alpha_1 - f\alpha_2)} (\mathbf{u}_1 - \mathbf{u}_2)$$

$$+ \rho_2 \alpha_2 \mathbf{g} + \frac{\rho_2}{(\rho_2 + f\rho_1)} (\rho_2 \mathbf{V}_2 + \rho_1 \mathbf{V}_3) . \quad [16]$$

The functional form of the $\mathbf{V}_3$ term does not influence the derivation below and need not be specified at this stage of the development. Substituting for the prime variables in [4], we have

$$\frac{\partial \rho_1(\alpha_1 \mathbf{u}_1 - f\alpha_2 \mathbf{u}_2)}{\partial t} + \nabla \cdot \left(\frac{\rho_1}{\alpha_1 - f\alpha_2}\right)(\alpha_1 \mathbf{u}_1 - f\alpha_2 \mathbf{u}_2)(\alpha_1 \mathbf{u}_1 - f\alpha_2 \mathbf{u}_2)$$

$$= -(\alpha_1 - f\alpha_2)\nabla p + \frac{\alpha_1(K_{12} + K_{13})}{(\alpha_1 - f\alpha_2)} (\mathbf{u}_2 - \mathbf{u}_1) + \rho_1(\alpha_1 - f\alpha_2)\mathbf{g} + \rho_1 \mathbf{V}_1 . \quad [17]$$

Equation [16] enables the elimination of the term $\partial \alpha_2 \mathbf{u}_2 / \partial t$ from [17]. Add and subtract

$\rho_1 \nabla \cdot \alpha_1 \mathbf{u}_1 \mathbf{u}_1$ from the result and rearrange to obtain

$$\rho_1\left[\frac{\partial \alpha_1 \mathbf{u}_1}{\partial t} + \nabla \cdot \alpha_1 \mathbf{u}_1 \mathbf{u}_1\right] = -\frac{+f\rho_1 + \rho_2(\alpha_1 - f\alpha_2)}{(\rho_2 + f\rho_1)} \nabla p - \frac{\rho_2 \alpha_1 (K_{12} + K_{13})}{(\rho_2 + f\rho_1)(\alpha_1 - f\alpha_2)}(\mathbf{u}_1 - \mathbf{u}_2) + \rho_1 \alpha_1 \mathbf{g} + \rho_1 V_1 + \frac{f\rho_1}{(\rho_2 + f\rho_1)}(\rho_2 V_2 + \rho_1 V_3) - f\rho_1 \nabla \cdot \left(\frac{\alpha_1 \alpha_2}{\alpha_1 - f\alpha_2}\right)(\mathbf{u}_2 - \mathbf{u}_1)(\mathbf{u}_2 - \mathbf{u}_1). \qquad [18]$$

At this stage of the derivation the effects of virtual mass are principally manifested in the coefficients of the pressure gradients and in the last term of[18]. For many purposes it is more convenient to rewrite the equations in a manner that removes the virtual mass effects from the coefficients of the pressure gradient terms and that isolates them in a way that allows direct comparison with previous formulations. To accomplish this rearrangement multiply[18] by $\rho_2\alpha_2$, [16] by $\rho_1\alpha_1$, subtract to eliminate the gravitational terms, and rearrange. The result is

$$0 = -\nabla p + \frac{\rho_1(\rho_2 + f\rho_1)}{(\rho_1 - \rho_2)(\alpha_1 - f\alpha_2)}\left(\frac{\partial \alpha_1 \mathbf{u}_1}{\partial t} - \frac{\alpha_1}{\alpha_2}\frac{\partial \alpha_2 \mathbf{u}_2}{\partial t} + \nabla \cdot \alpha_1 \mathbf{u}_1 \mathbf{u}_1 - \frac{\alpha_1}{\alpha_2}\nabla \cdot \alpha_2 \mathbf{u}_2 \mathbf{u}_2\right) + \frac{\alpha_1(\rho_1\alpha_1 + \rho_2\alpha_2)(K_{12} + K_{13})}{\alpha_2(\rho_1 - \rho_2)(\alpha_1 - f\alpha_2)^2}(\mathbf{u}_1 - \mathbf{u}_2) - \frac{\rho_1(\rho_2 + f\rho_1)}{(\rho_1 - \rho_2)(\alpha_1 - f\alpha_2)} V_1 + \frac{\rho_1}{\alpha_2(\rho_1 - \rho_2)}(\rho_2 V_2 + \rho_1 V_3) + \frac{f\rho_1(\rho_2 + f\rho_1)}{(\rho_1 - \rho_2)(\alpha_1 - f\alpha_2)} \nabla \cdot \left(\frac{\alpha_1\alpha_2}{\alpha_1 - f\alpha_2}\right)(\mathbf{u}_2 - \mathbf{u}_1)(\mathbf{u}_2 - \mathbf{u}_1). \qquad [19]$$

Multiply[19] by $-f\alpha_2(\rho_1 - \rho_2)/(\rho_2 + f\rho_1)$ and add the result to[18]

$$\rho_1\left[\frac{\partial \alpha_1 \mathbf{u}_1}{\partial t} + \nabla \cdot \alpha_1 \mathbf{u}_1 \mathbf{u}_1\right] = -\alpha_1 \nabla p - \frac{\alpha_1^2(K_{12} + K_{13})}{(\alpha_1 - f\alpha_2)^2}(\mathbf{u}_1 - \mathbf{u}_2) + \rho_1\alpha_1\mathbf{g} + \frac{\rho_1\alpha_1}{(\alpha_1 - f\alpha_2)} V_1 - \frac{f\rho_1\alpha_1}{(\alpha_1 - f\alpha_2)} \nabla \cdot \left(\frac{\alpha_1\alpha_2}{\alpha_1 - f\alpha_2}\right)(\mathbf{u}_2 - \mathbf{u}_1)(\mathbf{u}_2 - \mathbf{u}_1) - \frac{f\rho_1\alpha_2}{(\alpha_1 - f\alpha_2)} \times \left(\frac{\partial \alpha_1 \mathbf{u}_1}{\partial t} - \frac{\alpha_1}{\alpha_2}\frac{\partial \alpha_2 \mathbf{u}_2}{\partial t} + \nabla \cdot \alpha_1 \mathbf{u}_1 \mathbf{u}_1 - \frac{\alpha_1}{\alpha_2}\nabla \cdot \alpha_2 \mathbf{u}_2 \mathbf{u}_2\right). \qquad [20]$$

The two-field mass equations,[14] and [15], are used to remove the time derivatives of volume fraction from the last term of[20]. The equation expressing momentum conservation for field 1 then becomes

$$\rho_1\left[\frac{\partial \alpha_1 \mathbf{u}_1}{\partial t} + \nabla \cdot \alpha_1 \mathbf{u}_1 \mathbf{u}_1\right] - \frac{f\rho_1\alpha_1\alpha_2}{(\alpha_1 - f\alpha_2)}\left(\frac{\partial \mathbf{u}_2}{\partial t} - \frac{\partial \mathbf{u}_1}{\partial t}\right) = -\alpha_1\nabla p + \frac{\alpha_1^2(K_{12} + K_{13})}{(\alpha_1 - f\alpha_2)^2}(\mathbf{u}_2 - \mathbf{u}_1) + \rho_1\alpha_1\mathbf{g} + \frac{\rho_1\alpha_1}{(\alpha_1 - f\alpha_2)} V_1 - \frac{f\rho_1\alpha_1}{(\alpha_1 - f\alpha_2)} \nabla \cdot \left(\frac{\alpha_1\alpha_2}{\alpha_1 - f\alpha_2}\right)(\mathbf{u}_2 - \mathbf{u}_1)(\mathbf{u}_2 - \mathbf{u}_1) + \frac{f\rho_1\alpha_1\alpha_2}{(\alpha_1 - f\alpha_2)}(\mathbf{u}_2 \cdot \nabla \mathbf{u}_2 - \mathbf{u}_1 \cdot \nabla \mathbf{u}_1). \qquad [21]$$

The equation for field 2 is obtained by a similar series of steps. Equation[19] is multiplied by $f\alpha_2(\rho_1-\rho_2)/(\rho_2+f\rho_1)$ and the result is added is added to[16]. After rearranging, the expression is

$$\rho_2\left[\frac{\partial\alpha_2\boldsymbol{u}_2}{\partial t}+\nabla\cdot\alpha_2\boldsymbol{u}_2\boldsymbol{u}_2\right]+\frac{f\rho_1\alpha_1\alpha_2}{(\alpha_1-f\alpha_2)}\left(\frac{\partial\boldsymbol{u}_2}{\partial t}-\frac{\partial\boldsymbol{u}_1}{\partial t}\right)=-\alpha_2\nabla p+\frac{\alpha_1{}^2(K_{12}+K_{13})}{(\alpha_1-f\alpha_2)^2}(\boldsymbol{u}_1-\boldsymbol{u}_2)$$

$$+\rho_2\alpha_2\boldsymbol{g}-\frac{f\rho_1\alpha_2}{(\alpha_1-f\alpha_2)}\boldsymbol{V}_1+\rho_2\boldsymbol{V}_2+\rho_1\boldsymbol{V}_3+\frac{f^2\rho_1\alpha_2}{(\alpha_1-f\alpha_2)}\nabla\cdot\left(\frac{\alpha_1\alpha_2}{\alpha_1-f\alpha_2}\right)(\boldsymbol{u}_2-\boldsymbol{u}_1)(\boldsymbol{u}_2-\boldsymbol{u}_1)$$

$$-\frac{f\rho_1\alpha_1\alpha_2}{(\alpha_1-f\alpha_2)}(\boldsymbol{u}_2\cdot\nabla\boldsymbol{u}_2-\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_1). \qquad [22]$$

Alternately the last term on the right of[22] can be combined with the time derivative on the left to give

$$\frac{f\rho_1\alpha_1\alpha_2}{(\alpha_1-f\alpha_2)}\left[\left(\frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t}\right)_2-\left(\frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t}\right)_1\right],$$

in which the total derivatives (the Lagrangian time derivatives) describe the rate of change along the motion of each individual field. A similar alternative exists for[21].

DISCUSSION

We sum[21] and [22] and integrate them over an Eulerian control volume to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int(\rho_1\alpha_1\boldsymbol{u}_1+\rho_2\alpha_2\boldsymbol{u}_2)\,\mathrm{d}\tau+\int\hat{n}\cdot(\rho_1\alpha_1\boldsymbol{u}_1\boldsymbol{u}_1+\rho_2\alpha_2\boldsymbol{u}_2\boldsymbol{u}_2)\,\mathrm{d}a+\int\hat{n}\cdot(\hat{n}\hat{n})p\,\mathrm{d}a$$

$$+\int\hat{n}\cdot\boldsymbol{F}_V\,\mathrm{d}a+f\rho_1\int\hat{n}\cdot\left(\frac{\alpha_1\alpha_2}{\alpha_1-f\alpha_2}\right)(\boldsymbol{u}_2-\boldsymbol{u}_1)(\boldsymbol{u}_2-\boldsymbol{u}_1)\,\mathrm{d}a=\int(\rho_1\alpha_1+\rho_2\alpha_2)\boldsymbol{g}\,\mathrm{d}\tau. \qquad [23]$$

In[23] $\mathrm{d}\tau$ is an element of the control volume; $\mathrm{d}a$ is an element of the surface area enclosing the volume; and $\hat{n}$ is a unit vector that is normal to the surface. The equation is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{Total Momentum})+\int\hat{n}\cdot\boldsymbol{F}\,\mathrm{d}a=\int(\text{sources})\,\mathrm{d}\tau, \qquad [24]$$

where $\boldsymbol{F}$ represents the flux of momentum through the surface, showing that the virtual mass terms are internally conservative of momentum.

Under most circumstances the validity of the virtual mass concept for bubbles implies that $f\alpha_2$ is small compared to α_1, so that the otherwise vanishing denominators in our equations are of no concern.

This formulation has been incorporated in a two-dimensional computer code developed by Cook & Harlow (1983) to study the inception and evolution of bubbly von Karman vortex streets in two-phase flow, Cook & Harlow (to be submitted), in which it is shown that excellent agreement with experiments is obtained with this formulation.

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